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On the theory of solitary Rossby waves

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The evolution of long, finite amplitude Rossby waves in a horizontally sheared zonal current is studied. The wave evolution is described by the Korteweg–de Vries equation or the modified Korteweg–de Vries equation depending on the atmospheric stratification. In either case, the cross-stream modal structure of these waves is given by the long-wave limit of the neutral eigensolutions of the barotropic stability equation. Both non-singular and singular eigensolutions are considered and the appropriate analysis is developed to yield a uniformly valid description of the motion in the critical-layer region where the wave speed matches the flow velocity. The analysis demonstrates that coherent, propagating, eddy structures can exist in stable shear flows and that these eddies have peculiar interaction properties quite distinct from the traditional views of turbulent motion.

1. Introduction

Recently, a number of nonlinear equations describing the evolution of finite amplitude waves in dispersive media have been shown to possess exact analytical solutions whose single most distinctive feature is the existence of solitary waves. These waves, called solitons, emerge as permanent entities from quite general initial states of motion and are stable even to mutual interactions (see, for example, Scott, Chu & McLaughlin 1973). Perhaps the best known of these nonlinear evolution equations is the Korteweg–de Vries (KdV) equation, which was derived originally in the context of free-surface gravity waves propagating in shallow water.

In this paper we show that the amplitude of zonally propagating long planetary waves in an atmosphere with a horizontally sheared zonal flow obeys the KdV equation or the modified KdV (MKdV) equation depending on the existence and nature of the atmospheric density stratification. Long (1964) and Benney (1966) discussed long waves in a homogeneous atmosphere and obtained the KdV equation, but their analysis was limited to the case where the velocity shear was small compared with a basic uniform zonal motion and they gave no insight pertaining to the kinds of streamline-flow patterns accompanying these waves. Their limitation to a small shear superimposed on an order-one uniform flow avoided the special considerations required by the existence of a critical layer where the wave speed matches the zonal-flow velocity. Solitary Rossby waves were also studied by Larsen (1965) and Clarke (1971), but they, as well, avoided a discussion of the critical layer and did not provide any information about possible flow patterns. In the analysis which follows, we take the shear to be of order one and also consider the case when the atmosphere has a distributed, stable density stratification. Both generalizations require fundamentally different analytical considerations from the usual KdV theory. Specifically, a discussion of the dynamics in the vicinity of the critical latitude is required. The solution for the motion in the critical-layer region provides a description of the possible kinds of solitary-wave streamline patterns that can arise. It is found that, for an asymmetric shear layer such as $U(y) = \tanh y$, for example, two basic patterns emerge: one with closed, elliptically shaped streamlines and one with reversed flow both fore and aft of the wave centre. For other kinds of shear flows permitting multiple critical layers, a rich variety of eddying motion can accompany these waves.

The linear dispersion relation for the class of wave motions discussed here has the form

$$\omega = c_0 l - \gamma_0 l^3 + \ldots, \quad \psi \sim A \exp\{i(lx - \omega t)\}, \tag{1.1}$$

in the long-wave limit ($l \rightarrow 0$). Hence, if we consider a reference frame moving with the long-wave phase speed c_0, it is clear that the slow space and time scales must satisfy the relations $\xi = \epsilon^{\frac{1}{2}p}x$ and $\tau = \epsilon^{\frac{3}{2}p}t$, respectively, if the weak phase dispersion measured by γ_0 is to be balanced by the nonlinearity or amplitude dispersion. The integer value of p is consistent with the nonlinear term in the generalized evolution equation

$$A_\tau + A^p A_\xi + A_{\xi\xi\xi} = 0 \tag{1.2}$$

applicable for conservative systems of the type (1.1). The value of p emerges from the analysis. These comments provide the basis for the choice of the multiple scales used in the ensuing analysis.

2. Theoretical foundations

The mathematical basis for the theory presented herein is the quasi-geostrophic form of the potential-vorticity equation for an incompressible stratified fluid (cf. Pedlosky 1971):

$$\{\partial_{t'} + \Psi'_{y'}\partial_{x'} - \Psi'_{x'}\partial_{y'}\}[\partial^2_{x'x'} + \partial^2_{y'y'} + \partial_{z'}(f_0^2 N^{-2}\partial_{z'})]\Psi' + \beta'\Psi'_{x'} = 0. \tag{2.1}$$

Rossby's β-plane model has been used, in which the dynamical effects of planetary sphericity are retained only in the horizontal component of the Coriolis force and approximated as

$$\left.\begin{aligned} f = 2\Omega\sin\theta &= 2\Omega\sin\theta_0 + 2\Omega r_0^{-1}y'\cos\theta_0 = f_0 + \beta' y', \\ y' &= r_0(\theta - \theta_0). \end{aligned}\right\} \tag{2.2}$$

In these equations (x', y') are local Cartesian co-ordinates centred at latitude θ_0 and directed towards the east and north, respectively, and the velocities in these directions are $(\Psi'_{y'}, -\Psi'_{x'})$. The vertical co-ordinate is z' and, consistent with the order of approximation implicit in (2.1), the vertical velocity is given by

$$w' = f_0 N^{-2}\{\partial_{t'} + \Psi'_{y'}\partial_{x'} - \Psi'_{x'}\partial_{y'}\}\Psi'_{z'}, \tag{2.3}$$

where N is the Brunt–Väisälä frequency. The atmosphere is assumed to be confined between two fixed horizontal planes with a vertical separation distance D. Hence we require that $\Psi'_{z'}$ vanishes at $z' = 0, D$. The analysis carries through for free surfaces as well, but we do not include a general discussion of that case here. In the theoretical development we assume that latitudinal boundaries are also present, with a separation distance L. The latter restriction can be relaxed for certain models even within the context of the β-plane approximation and, in that case, we suppose that L is a characteristic measure of the width of the mean zonal shear flow. If the atmosphere is homogeneous, the derivatives with respect to z' are absent in (2.1).

It is convenient at this point to introduce dimensionless variables defined as

$$(x', y') = L(x, y), \quad z' = Dz, \quad t' = \frac{L}{U_0}t, \quad w' = \frac{D}{L}U_0\omega, \quad \Psi' = U_0 L\Psi, \quad \beta' = \frac{U_0}{L^2}\beta, \tag{2.4}$$

where U_0 is the scale of the zonal flow (the maximum of $U(y)$, say). The total stream function Ψ is considered to be composed of a disturbance, characterized by a non-dimensional amplitude parameter ϵ, superimposed on the zonal shear flow $U(y)$. Thus we write

$$\Psi(x, y, z, t) = \int_{y_c}^{y} \{U(y') - c_0\}\, dy' + \epsilon\psi(x, y, z, t), \tag{2.5}$$

where c_0 is a constant, which we later identify as the linear long-wave phase velocity of a Rossby wave in the shear flow, and y_c denotes the level where $U(y_c) = c_0$, provided that $U_{\min} < c_0 < U_{\max}$, but is otherwise arbitrary. With these definitions, (2.1) and (2.3) become

$$\{\partial_t + (U - c_0)\,\partial_x + \epsilon(\psi_y\,\partial_x - \psi_x\,\partial_y)\}\,[\partial_{xx}^2 + \partial_{yy}^2 + \partial_z(K^2\,\partial_z)]\,\psi + (\beta - U'')\,\psi_x = 0 \tag{2.6}$$

and

$$w = Ro\,K^2\epsilon\{\partial_t + (U - c_0)\,\partial_x + \epsilon(\psi_y\,\partial_x - \psi_x\,\partial_y)\}\,\psi_z. \tag{2.7}$$

Primes are now used to denote total derivatives with respect to y. The parameters appearing in these equations have the definitions

$$K = f_0 L/ND = L/L_r, \quad Ro = U_0/f_0 L. \tag{2.8}$$

The first parameter, a rotational internal Froude number, is assumed to be of order one. It compares the horizontal length scale L with the intrinsic scale L_r, called the Rossby radius of deformation. The effect of stratification is important only when L is of the order of L_r. The second parameter is the Rossby number, which has been assumed to be small in the derivation of (2.1). Equation (2.1) is, therefore, an equation describing the time evolution of a geostrophic flow and we now show that permanent or solitary waves can exist in such a flow whenever it contains persistent shearing motion.

2.1. *Stratified atmosphere* (N = *constant*)

In this section we derive the asymptotic solution of (2.6) for weakly nonlinear ($0 < \epsilon \ll 1$) long waves when the Brunt–Väisälä frequency is constant. To this end it is convenient to introduce the multiple-scale variables

$$\xi = \epsilon x, \quad \tau = \epsilon^3 t \tag{2.9}$$

and write the disturbance stream function as

$$\psi(x, y, z, t) = \psi^{(1)}(\xi, y, z, \tau) + \epsilon\psi^{(2)} + \epsilon^3\psi^{(3)} + \dots. \tag{2.10}$$

As will be seen shortly, this is the appropriate scaling to achieve a non-trivial balance between the effects of dispersion and nonlinear steepening for long waves. Defining the linear operator

$$\mathscr{L} \equiv \partial_\xi\{(U - c_0)\,(\partial_{yy}^2 + K^2\,\partial_{zz}^2) + (\beta - U'')\} \tag{2.11}$$

and substituting into (2.6) yields

$$\mathscr{L}\psi^{(1)}+\epsilon\{\mathscr{L}\psi^{(2)}+(\psi^{(1)}_y\partial_\xi-\psi^{(1)}_\xi\partial y)(\partial^2_{yy}+K^2\partial^2_{zz})\psi^{(1)}\}$$
$$+\epsilon^2\{\mathscr{L}\psi^{(3)}+(\psi^{(1)}_y\partial_\xi-\psi^{(1)}_\xi\partial y)(\partial^2_{yy}+K^2\partial^2_{zz})\psi^{(2)}+(\psi^{(2)}_y\partial_\xi-\psi^{(2)}_\xi\partial y)(\partial^2_{yy}+K^2\partial^2_{zz})\psi^{(1)}$$
$$+(U-c_0)\psi^{(1)}_{\xi\xi\xi}+(\partial^2_{yy}+K^2\partial^2_{zz})\psi^{(1)}_\tau\}+O(\epsilon^3)=0. \tag{2.12}$$

To leading order, the solution for a single internal-wave mode (n, say) satisfying the rigid-lid boundary conditions is

$$\psi^{(1)}=A_n(\xi,\tau)\,\phi_n(y)\cos n\pi z, \tag{2.13}$$

where $\phi_n(y)$ is determined by the solution of the equation

$$\left.\begin{aligned}\phi''_n-k^2_n\phi_n+\frac{\beta-U''}{U-c_{0n}}\phi_n=0,\quad k_n=n\pi K,\\ \text{with}\qquad \phi_n(y_1)=\phi_n(y_2)=0.\end{aligned}\right\} \tag{2.14}$$

The amplitude function $A_n(\xi,\tau)$ is arbitrary to this order. As (2.14) shows, the north–south modal structure of the wave is given by solutions to the barotropic stability equation in which the parameter k_n assumes the role of the wavenumber. Since we are explicitly seeking a long-wave solution, the wavenumber does not enter to this order. Equation (2.14) defines an eigenvalue problem for the real eigenvalue c_{0n} when $U(y)$, β and k_n are specified.

Before continuing with the theoretical development, we note that there are basically three different kinds of solution for $\phi_n(y)$ depending on the magnitude of c_{0n}. If c_{0n} is outside the range of $U(y)\,(c_{0n}<U_{\min}$ or $c_{0n}>U_{\max})$, solutions of (2.14) are well behaved for all y in $y_1<y<y_2$. We refer to this class of solutions as propagating neutral modes (PNM). The same holds for non-singular or regular neutral modes (RNM), for which $U_{\min}<c_{0n}<U_{\max}$ but

$$\lim_{y\to y_{cn}}B_n(y)=\lim_{y\to y_{cn}}\{(\beta-U'')/(U-c_{0n})\} \tag{2.15}$$

is finite. However, if $(\beta-U''(y_{cn}))\neq 0$ and $U_{\min}<c_{0n}<U_{\max}$, $\phi_n(y)$ is a singular neutral mode (SNM) and (2.13) is not uniformly valid for all y. In this case (2.13) is only an outer solution (for $y>y_{cn}$, say) and an inner solution (critical-layer analysis) is required to obtain connexion formulae across the critical layer at $y=y_{cn}$. Higher-order terms in the outer expansion for the RNM are also singular and require a separate critical-layer analysis. These latter cases are discussed separately in § 3.

For non-singular neutral modes we can continue the analysis and obtain

$$\psi^{(2)}=\tfrac{1}{4}A^2_n\{\phi^{(2,2)}_n(y)\cos 2n\pi z+\phi^{(2,0)}_n(y)\}, \tag{2.16}$$

where the $\phi^{(2,j)}_n$ are given by the inhomogeneous equation

$$\left.\begin{aligned}\phi^{(2,j)''}_n-4\delta_{2j}k^2_n\phi^{(2,j)}_n+\frac{\beta-U''}{U-c_{0n}}\phi^{(2,j)}_n=-\left(\frac{\beta-U''}{U-c_{0n}}\right)'\frac{\phi^2_n}{U-c_{0n}},\\ \text{with}\qquad \phi^{(2,j)}_n(y_1)=\phi^{(2,j)}_n(y_2)=0.\end{aligned}\right\} \tag{2.17}$$

There are no non-trivial solutions to the homogeneous equation and the particular solution is singular at $y=y_{cn}$ whenever $U_{\min}<c_{0n}<U_{\max}$, i.e. for both RN and SN modes. Note that the right-hand side vanishes when U is constant. It is for this reason

that solitary Rossby waves exist only if there is a horizontal shear in the zonal flow. Proceeding to the next order, one obtains the equation

$$\mathscr{L}\psi^{(3)} = (U-c_{0n})\{f_n^{(3)}(y)\,A_{n,\tau}+g_n^{(3)}(y)\,(A_n^3)_\xi+h_n^{(3)}(y)\,A_{n,\xi\xi\xi}\}\cos n\pi z$$
$$-(A_n^3)_\xi\frac{U-c_{0n}}{8}\left\{\frac{1}{3}\left(\frac{\beta-U''}{U-c_{0n}}\right)''\frac{\phi_n^3}{(U-c_{0n})^2}+\left(\frac{\beta-U''}{U-c_{0n}}\right)'\frac{\phi_n}{U-c_{0n}}\right.$$
$$\left.\times\left[\phi_n^{(2,2)}-\frac{1}{3}\frac{\phi_n^2U'}{(U-c_{0n})^2}\right]\right\}\cos 3n\pi z, \tag{2.18}$$

where

$$f_n^{(3)}(y) = \left(\frac{\beta-U''}{U-c_{0n}}\right)\frac{\phi_n}{U-c_{0n}}, \tag{2.19a}$$

$$g_n^{(3)}(y) = \frac{1}{8}\left(\frac{\beta-U''}{U-c_{0n}}\right)'\frac{\phi_n}{U-c_{0n}}\left\{\frac{U'\phi_n^2}{(U-c_{0n})^2}-(\phi_n^{(2,2)}+2\phi_n^{(2,0)})\right\}-\frac{1}{8}\left(\frac{\beta-U''}{U-c_{0n}}\right)''\frac{\phi_n^3}{(U-c_{0n})^2}, \tag{2.19b}$$

$$h_n^{(3)}(y) = -\phi_n. \tag{2.19c}$$

Restricting ourselves to modes without critical layers (i.e. PN modes), it is clear that a solution to (2.18) exists only if $A_n(\xi,\tau)$ evolves according to the modified Korteweg–de Vries (MKdV) equation

$$A_{n,\tau}+R_nA_n^2A_{n,\xi}+S_nA_{n,\xi\xi\xi} = 0, \tag{2.20}$$

where the coefficients R_n and S_n have the definitions

$$R_n = \frac{3}{I_n}\int_{y_1}^{y_2}\phi_n g_n^{(3)}\,dy, \tag{2.21a}$$

$$S_n = \frac{1}{I_n}\int_{y_1}^{y_2}\phi_n h_n^{(3)}\,dy = -\frac{1}{I_n}\int_{y_1}^{y_2}\phi_n^2\,dy, \tag{2.21b}$$

where
$$I_n = \int_{y_1}^{y_2}\phi_n f_n^{(3)}\,dy. \tag{2.21c}$$

For modes with critical layers ($U_{\min} < c_{0n} < U_{\max}$), the first integral does not exist, even as a principal value. However, the evolution equation (2.20) is still applicable with differently determined coefficients. The contribution to $\psi^{(3)}$ from the nth internal-wave mode, $\psi^{(3,1)}$ say, in any case can be written as

$$\psi^{(3,1)} = A_n^3\phi_1^{(3,1)}(y)\cos n\pi z+A_{n,\xi\xi}\,\phi_2^{(3,1)}(y)\cos n\pi z, \tag{2.22}$$

with
$$\phi_1^{(3,1)''}-k_n^2\,\phi_1^{(3,1)}+\frac{\beta-U''}{U-c_{0n}}\,\phi_1^{(3,1)} = g_n^{(3)}(y)-R_n\,f_n^{(3)}(y), \tag{2.23a}$$

$$\phi_2^{(3,1)''}-k_n^2\,\phi_2^{(3,1)}+\frac{\beta-U''}{U-c_{0n}}\,\phi_2^{(3,1)} = h_n^{(3)}(y)-S_n\,f_n^{(3)}(y). \tag{2.23b}$$

The solutions to these equations are

$$\phi_1^{(3,1)} = a_1\phi_a(y)+b_1\phi_b(y)+G_n^{(3)}(y)-R_nF_n^{(3)}(y), \tag{2.24a}$$

$$\phi_2^{(3,1)} = a_2\phi_a(y)+b_2\phi_b(y)+H_n^{(3)}(y)-S_nF_n^{(3)}(y), \tag{2.24b}$$

where ϕ_a and ϕ_b are the two linearly independent solutions to the homogeneous equation. Applying the boundary conditions yields

$$R_n = \frac{G_n^{(3)}(y_2)\,\phi_a(y_1) - G_n^{(3)}(y_1)\,\phi_a(y_2)}{F_n^{(3)}(y_2)\,\phi_a(y_1) - F_n^{(3)}(y_1)\,\phi_a(y_2)} \tag{2.25a}$$

and

$$S_n = \frac{H_n^{(3)}(y_2)\,\phi_a(y_1) - H_n^{(3)}(y_1)\,\phi_a(y_2)}{F_n^{(3)}(y_2)\,\phi_a(y_1) - F_n^{(3)}(y_1)\,\phi_a(y_2)}. \tag{2.25b}$$

These coefficients are well defined, even though the particular integrals $F_n^{(3)}$, $G_n^{(3)}$ and $H_n^{(3)}$ may be singular at $y = y_{cn}$, once the appropriate connexion relations across the critical layer have been established. Thus, even for the singular case, the MKdV equation remains the relevant evolution equation for the outer flow. It will be shown later that a systematic analysis of the critical-layer region provides no further restrictions on the evolution of the wave. It does yield, however, the connexion formulae across the critical level which are necessary for defining the eigenfunction $\phi_n(y)$ and computing the eigenvalues c_{0n}.

An alternative derivation of the coefficients R_n and S_n for RN and SN modes can be given which uses a modification of the solvability condition. Using the notation of (2.23a) and (2.24a) one can obtain the relation

$$[\phi_n'(G_n^{(3)} - R_n F_n^{(3)}) - \phi_n(G_n^{(3)} - R_n F_n^{(3)})']_{y_c-\delta}^{y_c+\delta} = \int_{y_1}^{y_c-\delta} \phi_n(g_n^{(3)} - R_n f_n^{(3)})\,dy + \int_{y_c+\delta}^{y_2} \phi_n(g_n^{(3)} - R_n f_n^{(3)})\,dy, \tag{2.25c}$$

where δ is a small constant ($O(\epsilon^{\frac{1}{2}})$, see § 3). This extension of the solvability condition was first given by Benney & Maslowe (1975b). The advantage over (2.25a) is that the complete solution for $\phi^{(3,1)}$ is not required, only the behaviour of the particular solution in the immediate vicinity of the critical level.

The MKdV equation admits soliton solutions whenever R_n is positive, so that the effect of the nonlinearity is to steepen the wave. Also, since the MKdV equation is invariant to a change in the sign of A_n, the solution for an isolated soliton is

$$\left.\begin{aligned} A_n(\xi,\tau) &= \pm\ \mathrm{sech}\,\{|R_n/6S_n|^{\frac{1}{2}}(\xi - c_{1n}\tau)\}, \quad R_n > 0,\\ c_{1n} &= \tfrac{1}{6}R_n\,\mathrm{sgn}\,S_n. \end{aligned}\right\} \tag{2.26}$$

Wadati (1973) has constructed the inverse scattering transform (IST) solution to the MKdV equation for appropriate initial data. His solution is a special case of a more general technique developed by Ablowitz *et al.* (1974). Cnoidal-wave solutions of (2.20) are also possible and they can be modulationally stable or unstable depending on the choice of certain wave parameters (cf. Driscoll & O'Neil 1976), but we shall not consider these waves any further here.

2.2. *Homogeneous atmosphere*

The case when the atmosphere is homogeneous, so that there is no variation of the flow quantities in the vertical, is studied in this section. The effect of divergence (i.e. a free upper surface) is included in the discussion since it is straightforward to do so. For this atmospheric model the governing equation is

$$\{\partial_t + (U - c_0)\,\partial_x + \epsilon(\psi_y\,\partial_x - \psi_x\,\partial_y)\}\,(\partial_{xx}^2 + \partial_{yy}^2 - k_0^2)\,\psi + (\beta + k_0^2\,U - U'')\,\psi_x = 0. \tag{2.27}$$

The divergence parameter k_0^2 has the definition

$$k_0^2 = f_0^2 L^2 / g^* D, \tag{2.28}$$

where g^* is the reduced gravity $(g\Delta\rho/\rho)$ applicable to the free surface, which is located a distance D above a flat bottom boundary. In the analysis which follows we take k_0^2 to be an order-one parameter, but the non-divergent limit $k_0^2 = 0$ is uniform and can be taken without further modification.

The appropriate long-wave scaling for this case has

$$\xi = \epsilon^{\frac{1}{2}} x, \quad \tau = \epsilon^{\frac{3}{2}} t. \tag{2.29}$$

Although different definitions are given for the scaled variables, the same notation is used as in the previous subsection. This is the scaling required to balance the effects of dispersion and quadratic nonlinearity leading to the familiar KdV theory. If the disturbance stream function is then written as a regular perturbation series in the small amplitude parameter ϵ,

$$\psi(x,y,t) = \psi^{(1)}(\xi,y,\tau) + \epsilon\psi^{(2)} + \ldots = A(\xi,\tau)\,\phi(y) + \epsilon\psi^{(2)} + \ldots, \tag{2.30}$$

one finds that, to leading order, the modal structure of a long wave is prescribed by the eigenvalue problem

$$\phi'' - k_0^2\phi + \frac{\beta + k_0^2 U - U''}{U - c_0}\phi = 0, \quad \phi(y_1) = \phi(y_2) = 0. \tag{2.31}$$

The inhomogeneous equation for $\psi^{(2)}$ then becomes

$$\partial_\xi\{(U-c_0)(\partial^2_{yy} - k_0^2) + (\beta + k_0^2 U - U'')\}\psi^{(2)}$$
$$= A_\tau \frac{\beta + k_0^2 U - U''}{U - c_0}\phi - AA_\xi \left(\frac{\beta + k_0^2 U - U''}{U - c_0}\right)' \phi^2 - A_{\xi\xi\xi}(U - c_0)\,\phi. \tag{2.32}$$

A separable solution for $\psi^{(2)}$ is possible only if $A(\xi,\tau)$ evolves according to the familiar KdV equation

$$A_\tau + R_0 A A_\xi + S_0 A_{\xi\xi\xi} = 0. \tag{2.33}$$

The values of the coefficients R_0 and S_0 for non-singular neutral-mode solutions to (2.31) are determined, by invoking the solvability condition, to be

$$R_0 = \int_{y_1}^{y_2} \left(\frac{\beta + k_0^2 U - U''}{U - c_0}\right)' \frac{\phi^3}{U - c_0}\,dy \Bigg/ \int_{y_1}^{y_2} \left(\frac{\beta + k_0^2 U - U''}{U - c_0}\right) \frac{\phi^2}{U - c_0}\,dy, \tag{2.34a}$$

$$S_0 = \int_{y_1}^{y_2} \phi^2\,dy \Bigg/ \int_{y_1}^{y_2} \left(\frac{\beta + k_0^2 U - U''}{U - c_0}\right) \frac{\phi^2}{U - c_0}\,dy. \tag{2.34b}$$

These integrals exist as principal values when $U_{\min} < c_0 < U_{\max}$ provided that

$$B_0(y) = (\beta + k_0^2 U - U'')/(U - c_0) \tag{2.35}$$

is a regular function for all $y_1 < y < y_2$, i.e. for RN modes. The singular case requires the same critical-layer consideration as was mentioned for the stratified atmosphere in the previous subsection.

Long (1964) and Benney (1966) derived the KdV equation describing the evolution of long waves in a homogeneous atmosphere also, but their analysis assumed *a priori* that the sheared current was small compared with an order-one uniform mean flow.

In the present case, no such restrictions are imposed and some examples of the streamline patterns for an isolated soliton when the shear is of order one are presented in § 4. The solution of (2.33) for an isolated soliton is

$$\left.\begin{aligned} A(\xi,\tau) &= \operatorname{sgn}(R_0 S_0)\operatorname{sech}^2\{|R_0/12S_0|^{\frac{1}{2}}(\xi - c_1\tau)\},\\ c_1 &= -\tfrac{1}{3}|R_0|\operatorname{sgn} S_0. \end{aligned}\right\} \tag{2.36}$$

Only one sign of the amplitude function is acceptable now in contrast to the MKdV case (2.26) and, since the sign depends on the coefficients R_0 and S_0, it may change for different positions on the eigenlocus defined by (2.31).

It is interesting to note that the time and length scales and the evolution equations are fundamentally different for the two cases of a stratified atmosphere with a constant Brunt–Väisälä frequency and a homogeneous atmosphere. These two seemingly disparate limits can be most easily unified by considering the more general case of arbitrary stratification. Then the linear solution to (2.6) can be expressed as

$$\psi^{(1)} = \sum_n A_n(x,t)\,\phi_n(y)\,Z_n(z), \tag{2.37}$$

where $Z_n(z)$ satisfies the following eigenvalue problem for k_n^2:

$$(K^2(z)\,Z_n')' + k_n^2 Z_n = 0, \quad Z_n(0) = Z_n(1) = 0. \tag{2.38}$$

By writing the vertical structure at each order in the solution in terms of the complete set Z_n, one can obtain an evolution equation with both quadratic and cubic nonlinearities. The case of constant stratification is special in that the coupling coefficient for the quadratic term vanishes identically. The homogeneous case has $k_n^2 = 0$ and $Z_n = 1$. The general case (2.37) is discussed by Redekopp & Weidman (1977) in the context of interacting solitary waves with different linear phase speeds described by coupled systems of KdV equations.

3. Nonlinear critical-layer analysis

In this section we develop the inner (critical-layer) expansion so that a solution which is uniformly valid for all y can be constructed when the linear phase speed c_{0n} lies within the range of $U(y)$. Both singular neutral modes (SNM), for which the linear eigenvalue equation (2.14) is singular, and regular neutral modes (RNM), for which the eigenvalue equation is regular but higher-order terms in the outer expansion are singular, are considered. The requisite critical-layer analysis for these modes involves a generalization of the nonlinear critical-layer theory developed by Benney & Bergeron (1969).

During the early stages of this work, a paper by Benney & Maslowe (1975*a*) appeared in which they discuss the nonlinear critical-layer theory appropriate to the time evolution of finite amplitude waves in a parallel shear flow, having in mind the early stages of instability and transition of such flows. Their analysis and that which follows have common features, such as the form of the inner expansion, etc. Nevertheless, there are several important differences and extensions to the theory which are worthy of a separate discussion here. Specifically, the application to long waves, the modifications necessary in the application of the viscous secularity condition at higher orders, and the solution for RN modes are discussed.

Since the analysis presented in this section is lengthy, tedious and non-trivial, we summarize the essential details at the outset. The nonlinear critical-layer analysis applies when $\epsilon \gg R^{-\frac{2}{3}}$, where R is the Reynolds number, in the MKdV case and requires $\epsilon \gg R^{-\frac{1}{3}}$ in the KdV case. This restriction ensures that the advection of vorticity dominates viscous diffusion in the layer. We conclude for this case that there is no phase change in the outer solution across the critical layer and, hence, that the wave evolution is determined entirely by the outer flow. The analysis reveals that closed-streamline regions exist within which the potential vorticity is constant (essentially zero). The total stream function defining the flow pattern in the critical layer is given, to leading order, by†

$$\Psi = \epsilon S = \tfrac{1}{2}U_c'(y-y_c)^2 + \epsilon A(\xi,\tau)\,\phi_c \cos n\pi z, \tag{3.1}$$

where $\phi_c = \phi(y_c)$. This equation applies for both the MKdV and the KdV theory. The dividing-streamline shape can be computed directly for an isolated soliton when $A(\xi,\tau)$ is given either by (2.26) or by (2.36). Assuming that U_c' and ϕ_c are positive and noting that $0 < |A(\xi,\tau)| < 1$, we obtain

$$(y-y_c)_{\mathrm{dsl}} = \pm\,\epsilon^{\frac{1}{2}}\{2\phi_c U_c'^{-1}(S_c(z) - A\cos n\pi z)\}^{\frac{1}{2}}, \tag{3.2}$$

where $S_c(z)$ is the critical value of S/ϕ_c on the dividing streamline (dsl). When $A\cos n\pi z$ is negative, S_c is zero and the latitudinal variation of the dividing streamline is proportional to $\operatorname{sech}^{\frac{1}{2}}\theta$ for an MKdV soliton and $\operatorname{sech}\theta$ for a KdV soliton. When $A\cos n\pi z$ is positive, $S_c = |\cos n\pi z|$ and the location of the dividing streamline is proportional to $(1-\operatorname{sech}\theta)^{\frac{1}{2}}$ for an MKdV soliton and $\tanh\theta$ for a KdV soliton. We call the first type an E-soliton (envelope or elevation) and the latter type a D-soliton (diverting or depression). The two types of soliton are depicted schematically in figure 1. These types can be interchanged if U_c' and/or ϕ_c change sign and both types can appear simultaneously in shear flows with two or more distinct critical levels (e.g. a jet). The modifications in the analysis for these conditions are straightforward and we assume that U_c' and ϕ_c are both positive in the remainder of this section.

3.1. *Singular neutral modes*

As a preliminary to discussing the inner solution for SN modes, we recall the two linearly independent Frobenius solutions of the eigenvalue equation (2.14) valid in the vicinity of the critical point $y = y_c$. The regular solution is

$$\begin{aligned}\phi_a(y) &= (y-y_c) - \left(\frac{\beta - U_c''}{2U_c'}\right)(y-y_c)^2 + \left[\frac{k^2}{6} + \frac{U_c'''}{6U_c'} + \frac{\beta(\beta-U_c'')}{12(U_c')^2}\right](y-y_c)^3 + \dots \\ &= \sum_{n=1}^{\infty} a_n (y-y_c)^n \end{aligned} \tag{3.3a}$$

and the singular solution is

$$\begin{aligned}\phi_b(y) = 1 &+ \left[\frac{k^2}{2} + \frac{U_c'''}{2U_c'} + \frac{1}{4}\frac{U_c''}{U_c'}\frac{\beta - U_c''}{U_c'} - \frac{3}{4}\left(\frac{\beta-U_c''}{U_c'}\right)^2\right](y-y_c)^2 + \dots \\ &- \frac{\beta - U_c''}{U_c'}\phi_a(y)\log|y-y_c| = \sum_{n=0}^{\infty} b_n(y-y_c)^n + b_{00}\phi_a(y)\log|y-y_c|. \end{aligned} \tag{3.3b}$$

† Henceforth we omit the subscript n denoting the internal-wave mode number from $A(\xi,\tau)$, $\phi(y)$, y_c, etc., for purposes of convenience and to simplify lengthy expressions.

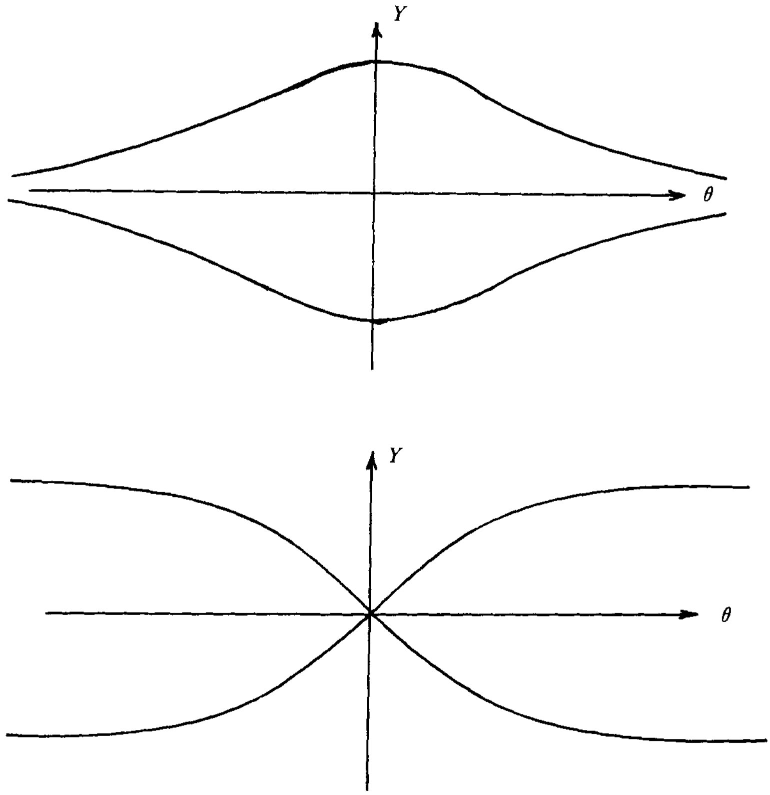

FIGURE 1. Schematic diagram of the dividing streamline for an E- and a D-soliton.

We choose to write the singular solution using $\log|y-y_c|$ as opposed to $\log(y-y_c)$ so that all expressions are real. The outer solution for $y > y_c$ is then written as

$$\psi_+^{(1)} = A(\xi,\tau)\,[\alpha\phi_a(y)+\phi_b(y)]\cos n\pi z \tag{3.4a}$$

and that for $y < y_c$ is written as

$$\psi_-^{(1)} = A(\xi,\tau)\,[\gamma\phi_a(y)+\rho\phi_b(y)]\cos n\pi z + D(\xi,\tau)\,[\kappa\phi_a(y)+\mu\phi_b(y)]\cos n\pi z. \tag{3.4b}$$

The appropriate branch of the logarithm for $y < y_c$, contained in the constants in (3.4b), is to be determined by matching the two expressions across the critical layer. From the analysis in §2.1, we know that $A(\xi,\tau)$ is prescribed by the MKdV equation, but we allow for the possibility that, in general, there may be a phase shift in the outer solution required by the matching across the critical level. We do not give the evolution equation for $D(\xi,\tau)$ since, as will be seen shortly, it is not needed. We also record the behaviour of the particular integral $\phi^{(2,j)}$ from (2.17) valid near the critical point, namely

$$\left.\begin{aligned}\phi^{(2,j)}(y) &= (\log|y-y_c|)^2\left\{\frac{(\beta-U_c'')^2}{(U_c')^3}+O(y-y_c)\right\}\\ &\quad+\log|y-y_c|\left\{\left[\frac{5}{2}\frac{(\beta-U_c'')^2}{(U_c')^3}-\frac{U_c''}{U_c'}\frac{\beta-U_c'''}{(U_c')^2}\right]+O(y-y_c)\right\}+\frac{\beta-U_c''}{2(U_c')^2}(y-y_c)^{-1}+O(1)\\ &= (\log|y-y_c|)^2\sum_{n=0}^{\infty}p_n(y-y_c)^n+\log|y-y_c|\sum_{n=0}^{\infty}q_n(y-y_c)^n+\sum_{n=-1}^{\infty}r_n(y-y_c)^n.\end{aligned}\right\} \tag{3.5}$$

The particular solutions become progressively more singular near the critical point as one goes to higher orders in the outer expansion. For example, the form of the particular solution for $\phi_1^{(3,1)}$ is

$$\phi_1^{(3,1)}(y) = (\log|y-y_c|)^3 \sum_{n=0}^{\infty} t_n(y-y_c)^n + (\log|y-y_c|)^2 \sum_{n=0}^{\infty} u_n(y-y_c)^n + \log|y-y_c| \sum_{n=0}^{\infty} v_n(y-y_c)^n + \sum_{n=-2}^{\infty} w_n(y-y_c)^n. \quad (3.6)$$

Expressions (3.3), (3.5), (3.6), etc., determine the scale of the critical layer and also the form of the inner expansion. As shown by Benney & Bergeron, a balance between the linear and the nonlinear advective terms in the critical-layer region, which also permits a consistent matching with the outer solution (3.4), is achieved by the stretching transformation

$$y-y_c = \epsilon^{\frac{1}{2}} Y. \quad (3.7)$$

Then, writing the outer expansion in terms of the inner variable Y, we obtain, for the total stream function, that

$$\begin{aligned}\lim_{Y\to\infty} \Psi = \epsilon\Bigg\{&\left(\frac{U_c'}{2} Y^2 + A\cos n\pi z\right) + \epsilon^{\frac{1}{2}}\log\epsilon\left(\frac{b_{00}}{2} YA\cos n\pi z\right)\\ &+\epsilon^{\frac{1}{2}}\left[\frac{U_c''}{3!}Y^3 + Y(\alpha + b_{00}\log|Y|)A\cos n\pi z + \frac{r_{-1}}{Y}\frac{A^2}{4}(\cos 2n\pi z+1)+\dots\right]\\ &+\epsilon\log^2\epsilon\left[\frac{p_0A^2}{16}(\cos 2n\pi z+1)\right]\\ &+\epsilon\log\epsilon\left[\frac{b_{00}a_2}{2}Y^2A\cos n\pi z + \left(p_0\log|Y| + \frac{q_0}{2}\right)\frac{A^2}{4}(\cos 2n\pi z+1)+\dots\right]\\ &+\epsilon\Bigg[\frac{U_c'''}{4!}Y^4 + Y^2(b_2 + b_{00}a_2\log|Y| + \alpha a_2)A\cos n\pi z\\ &+(r_0+p_0\log^2|Y| + q_0\log|Y|)\frac{A^2}{4}(\cos 2n\pi z+1)+\dots\Bigg]\\ &+O(\epsilon^{\frac{3}{2}}\log^3\epsilon, \epsilon^{\frac{3}{2}}\log^2\epsilon, \epsilon^{\frac{3}{2}}\log\epsilon, \epsilon^{\frac{3}{2}},\dots)\Bigg\} \end{aligned} \quad (3.8a)$$

and that

$$\begin{aligned}\lim_{Y\to-\infty} \Psi = \epsilon\Bigg\{&\left[\frac{U_c'}{2}Y^2 + (\rho A+\mu D)\cos n\pi z\right] + \epsilon^{\frac{1}{2}}\log\epsilon\left[\frac{b_{00}}{2}Y(\rho A+\mu D)\cos n\pi z\right]\\ &+\epsilon^{\frac{1}{2}}[Y(\gamma+\rho b_{00}\log|Y|)A\cos n\pi z + Y(\kappa+\mu b_{00}\log|Y|)D\cos n\pi z+\dots]+\dots\Bigg\}. \end{aligned} \quad (3.8b)$$

Only the terms necessary to establish the connexion between (3.4a, b) across the critical level are included in the latter expression. The first expression is written in such a way that the interpretation as $Y\to-\infty$ can also be made unambiguously. Higher-order matching conditions can be computed in a straightforward, albeit tedious, manner. The results show that the inner expansion proceeds as

$$\Psi = \epsilon\{\Psi^{(0)}(\xi, Y, z, \tau) + \epsilon^{\frac{1}{2}}\log\epsilon\Psi^{(1)} + \epsilon^{\frac{1}{2}}\Psi^{(2)} + \epsilon\log^2\epsilon\Psi^{(3)} + \epsilon\log\epsilon\Psi^{(4)} + \epsilon\Psi^{(5)} + \epsilon^{\frac{3}{2}}\log^3\epsilon\Psi^{(6)} + \epsilon^{\frac{3}{2}}\log^2\epsilon\Psi^{(7)} + \epsilon^{\frac{3}{2}}\log\epsilon\Psi^{(8)} + \epsilon^{\frac{3}{2}}\Psi^{(9)} + \dots\}. \quad (3.9)$$

Further terms in the expansion follow the indicated pattern.

The governing equation describing the motion in the critical-layer region is (2.6) written in terms of the inner variable Y:

$$\{\epsilon^{\frac{1}{2}}\partial_\tau+(\Psi_Y\partial_\xi-\Psi_\xi\partial_Y)\}(\partial^2_{YY}+\epsilon K^2\partial^2_{zz}+\epsilon^3\partial^2_{\xi\xi})\Psi+\epsilon^{\frac{1}{2}}\beta\Psi_\xi=\epsilon\lambda\Psi_{YYYY}. \quad (3.10)$$

The term on the right-hand side is the leading viscous term and the coefficient λ has the definition

$$\lambda=(R\epsilon^{\frac{1}{2}})^{-1}. \quad (3.11)$$

We are interested in the situation where the value of λ is always much smaller than unity, and hence we suppose that each of the terms in the expansion (3.9) possesses a regular perturbation in terms of λ. This implies, therefore, that the Reynolds number R bears a unique relation to the amplitude parameter ϵ. We note that the viscous parameter λ is different from that defined by Benney & Bergeron or Benney & Maslowe because of the long-wave approximation used here. In fact, if we were to formulate the analysis in terms of the KdV theory instead of the MKdV case, the parameter λ would have the definition $(R\epsilon^2)^{-1}$. Viscous effects are included because recourse to their ultimate influence on the motion is necessary in order to render the nonlinear critical-layer solution unique. It should not be surprising that the role of viscosity cannot be avoided whenever a unique solution for a flow having closed-streamline motions is desired.

The equation for the leading-order behaviour in the critical layer is nonlinear and is given by

$$(\Psi^{(0)}_Y\partial_\xi-\Psi^{(0)}_\xi\partial_Y)\Psi^{(0)}_{YY}=\lambda\Psi^{(0)}_{YYYY}. \quad (3.12)$$

Recalling that λ is indeed small, we suppose that a solution of the form

$$\Psi^{(0)}=\Psi^{(0,0)}+\lambda\Psi^{(0,1)}+\dots \quad (3.13)$$

must exist. This will always be true except when local gradients of the velocity field specified by $\Psi^{(0,0)}$ are large. It is useful at this stage to transform the independent variables in the manner

$$(\xi,Y,\tau,z)\to(X=\xi,\ S=\Psi^{(0,0)},\ T=\tau,\ Z=z) \quad (3.14)$$

to facilitate the integration of the equations for the sequence of functions in the inner expansion. Then, using the matching condition (3.8a) for $y>y_c$, one finds that

$$\tfrac{1}{2}S^2_Y=F^{(0)}(S,T,Z)-U'_cA\cos n\pi z, \quad (3.15)$$

where $F^{(0)}$ is an arbitrary function. Its value is made unique by examining the linear equation for $\Psi^{(0,1)}$ and requiring that $\Psi^{(0,1)}$ is not secular, i.e. that it admits a solution having the periodicity consistent with the outer flow (see discussion in the next paragraph). The result obtained after applying this condition is

$$\Psi^{(0)}=S=\tfrac{1}{2}U'_cY^2+A(\xi,\tau)\cos n\pi z. \quad (3.16)$$

This solution is valid throughout the critical layer and hence, by imposing the matching condition (3.8b), yields $\rho=1$ and $\mu=0$. The solution (3.16) reveals the dominant flow structure in the critical layer as explained in the introduction to this section. Proceeding in a similar fashion to the next order we obtain

$$\Psi^{(1)}=\tfrac{1}{2}b_{00}YA(\xi,\tau)\cos n\pi z, \quad (3.17)$$

which satisfies the matching condition identically with the previous choice for ρ and μ. The viscous corrections for both of these solutions are identically zero.

The analysis is more complicated at the next order and, since the remaining constants in (3.4*b*) are determined at this stage, a few more details are presented. The equation for $\Psi^{(2)}$ can be written in the form

$$(S_Y\,\partial_\xi - S_\xi\,\partial_Y)\,\Gamma = \lambda\Gamma_{YY}, \tag{3.18}$$

where

$$\Gamma = \Psi^{(2)}_{YY} - \beta Y. \tag{3.19}$$

Writing the function Γ, which, correct to this order, is the negative of the potential vorticity in the critical layer, in a power series as

$$\Gamma = \Gamma^{(0)} + \lambda\Gamma^{(1)} + \ldots, \tag{3.20}$$

we obtain

$$\Gamma^{(0)} = F^{(2,0)}(S, T, Z). \tag{3.21}$$

The equation for $\Gamma^{(1)}$ can be written in the form

$$\Gamma^{(1)}_X = S_Y\,F^{(2,0)}_{SS} + S_{YY}\,S_Y^{-1}F^{(2,0)}_S = \{\operatorname{sgn} Y[2U_c'(S - A\cos n\pi z)]^{\frac{1}{2}}\,F^{(2,0)}_S\}_S, \tag{3.22}$$

after using (3.15) to obtain

$$S_Y = \operatorname{sgn} Y[2U_c'(S - A\cos n\pi z)], \quad S > S_c(T, Z). \tag{3.23}$$

At this stage we invoke the viscous secularity condition, by which we require the viscous correction to the flow in the critical layer to exhibit the same periodicity as that forced by the outer flow. For a cnoidal wave train with wavelength $2\pi/\Delta$, this condition takes the form

$$\int_0^{2\pi/\Delta} \Gamma^{(1)}_X\,dX = 0. \tag{3.24}$$

However, in order also to include the initial-value problem on compact support and the solitary-wave limit $(\Delta \to 0)$ of a cnoidal wave train, we write the secularity condition as

$$\frac{\Delta}{2\pi}\int_{-\pi/\Delta}^{\pi/\Delta} \Gamma^{(1)}_X\,dX = \left\{F^{(2,0)}_S(\operatorname{sgn} Y)\frac{\Delta}{2\pi}\int_{-\pi/\Delta}^{\pi/\Delta}[2U_c'(S - A\cos n\pi z)]^{\frac{1}{2}}\,dX\right\}_S = 0, \quad S > S_c. \tag{3.25}$$

The 'wavenumber' Δ may be a slowly varying function of time. Equation (3.25) then shows that the unknown function $F^{(2,0)}$, for $S > S_c$, must be of the form

$$F^{(2,0)} = M(T, Z)\{I(S, T, Z) + (2S/U_c')^{\frac{1}{2}}\} + N(T, Z), \tag{3.26a}$$

where

$$I(S, T, Z) = \int_{S_c}^{S}\left\{\left[\frac{\Delta}{2\pi}\int_{-\pi/\Delta}^{\pi/\Delta}[2U_c'(\zeta - A\cos n\pi z)]^{\frac{1}{2}}\,dX\right]^{-1} - (2U_c'\zeta)^{-\frac{1}{2}}\right\}d\zeta. \tag{3.26b}$$

Applying the matching condition causes us to choose

$$M = (\operatorname{sgn} Y)\,U_c'b_{00}, \quad N = -I(S = \infty)\,M. \tag{3.27}$$

Within the dividing streamline ($S < S_c$) the potential vorticity must be constant if the solution is to be non-singular (cf. Batchelor 1956). Thus, when $S < S_c$,

$$F^{(2,0)} = N_0, \tag{3.28}$$

from which one finds that the vorticity cannot be continuous across the dividing streamline and that thin transition layers, within which the viscous terms enter to leading order, exist along these streamlines. Benney & Bergeron give the solution for the viscous layers when the dividing streamlines are sinusoidal and their analysis can be extended immediately to the present case.

We compute the velocity field by returning to (3.19) and integrating (3.26) and (3.28) once, which yields

$$\Psi_Y^{(2,0)} = \begin{cases} \operatorname{sgn} Y \int_{S_c}^{S} \frac{F^{(2,0)}(\zeta, T, Z)\, d\zeta}{[2U_c'(\zeta - A\cos n\pi Z)]^{\frac{1}{2}}} + \frac{\beta}{U_c'}(S - S_c) + V(T, X, Z), & S > S_c, \\ N_0 Y + \tfrac{1}{2}\beta Y^2 + V_0(T, X, Z), & S < S_c. \end{cases} \tag{3.29}$$

Then, after obtaining the asymptotic behaviour and also insisting that the velocity be continuous across the dividing streamlines, one finds by matching that

$$\gamma = \alpha, \quad \kappa = 0, \quad N_0 = 0. \tag{3.30}$$

There is no phase shift across the critical layer and the potential vorticity to this order is zero within the recirculating or closed-streamline regions. The leading-order behaviour in both the outer and the critical-layer flow is now determined.

The viscous correction $\Gamma^{(1)}$ in the critical layer can be computed straightaway to be

$$\Gamma^{(1)} = F^{(2,1)}(S, T, Z) + \int_{-\infty}^{X} (S_Y F_S^{(2,0)})_S \, dX, \tag{3.31}$$

but the unknown function $F^{(2,1)}$ must be zero since there is no viscous term in the outer flow to match to at this order. The first non-vanishing viscous term which must be matched to the outer flow, for the MKdV theory, enters at $O(\epsilon^3 \log \epsilon)$.

Higher-order terms in the inner expansion can be computed, in principle, by following the above procedure. For example, the next few terms have the form

$$\Psi^{(3)} = (b_{00}^2/16U_c')A^2(\cos 2n\pi z + 1), \tag{3.32}$$

$$\Psi^{(4,0)} = F^{(4,0)}(S, T, Z) + F_S^{(2,0)}\Psi^{(1)}, \tag{3.33}$$

$$\Psi^{(5,0)} = F^{(5,0)}(S, T, Z) + F_S^{(2,0)}\Psi^{(2)} + k^2 A \cos n\pi z, \quad \text{etc.} \tag{3.34}$$

At each stage the secularity condition and matching are used to determine the unknown functions. It becomes necessary, however, to introduce another (viscous) time scale in the critical layer if the viscous $O(\lambda)$ correction at each order is to remain uniformly valid. We choose to terminate the discussion of SN modes at this point, but shall exhibit the need for these modifications to the secularity condition in the next subsection, where we discuss the critical-layer solution for RN modes.

3.2. *Regular neutral modes*

When considering RN modes, the singularities in the outer expansion are postponed to higher-order terms compared with SN modes. The leading-order eigenfunction $\phi(y)$ is regular for all y and the next-order solutions [see (2.17) and (2.23)] have the form

$$\begin{aligned} \phi_{(y)}^{(2,j)} = {} & \log|y - y_c| \left\{ \frac{1}{2} \frac{\phi_c^2}{U_c'} \frac{U_c'''}{U_c'} \left(\frac{U_c^{\text{iv}}}{U_c'''} - \frac{U_c''}{U_c'} \right)(y - y_c) + \dots \right\} \\ & + \frac{\phi_c^2}{12U_c'} \frac{U_c'''}{U_c'} \left\{ \left[\frac{U_c^{\text{v}}}{U_c'''} - \frac{U_c''''}{U_c'} + \frac{3}{2}\left(2\frac{U_c''}{U_c'} - \frac{U_c^{\text{iv}}}{U_c'''} \right) \frac{U_c''}{U_c'} \right] \right. \\ & \left. + 6 \left(\frac{U_c^{\text{iv}}}{U_c'''} - \frac{U_c''}{U_c'} \right) \left(\frac{\phi_c'}{\phi_c} - \frac{U_c''}{4U_c'} \right) \right\} (y - y_c)^2 + \dots \\ = {} & \log|y - y_c| \sum_{n=1}^{\infty} P_n (y - y_c)^n + \sum_{n=2}^{\infty} Q_n (y - y_c)^n, \end{aligned} \tag{3.35}$$

$$\phi_1^{(3,1)} = (\log|y - y_c|)^2 \sum_{n=2}^{\infty} U_n (y - y_c)^n + \log|y - y_c| \sum_{n=0}^{\infty} V_n (y - y_c)^n + \sum_{n=-1}^{\infty} W_n (y - y_c)^n, \quad \text{etc.} \tag{3.36}$$

These results show that the critical-layer expansion for these modes must proceed as follows:

$$\Psi = \epsilon\{\Psi^{(0)}+\epsilon^{\frac{1}{2}}\Psi^{(1)}+\epsilon\Psi^{(2)}+\epsilon^{\frac{3}{2}}\log\epsilon\Psi^{(3)}+\epsilon^{\frac{3}{2}}\Psi^{(4)}+\epsilon^{2}\log\epsilon\Psi^{(5)}+\epsilon^{2}\Psi^{(6)} + \epsilon^{\frac{5}{2}}\log\epsilon\Psi^{(7)}+\epsilon^{\frac{5}{2}}\Psi^{(8)}+\epsilon^{3}\log^{2}\epsilon\Psi^{(9)}+\epsilon^{3}\log\epsilon\Psi^{(10)}+\epsilon^{3}\Psi^{(11)}+\ldots\}. \quad (3.37)$$

The matching condition for the functions appearing in this expansion is evaluated to be

$$\lim_{|Y|\to\infty}\Psi = \epsilon\left\{\left(\frac{U_c'}{2}Y^2+\phi_c A\cos n\pi z\right)+\epsilon^{\frac{1}{2}}\left(\frac{U_c''}{6}Y^3+\phi_c' YA\cos n\pi z\right) + \epsilon^{2}\left(\frac{U_c'''}{4!}Y^4+\frac{\phi_c''}{2}Y^2A\cos n\pi z\right)+\epsilon^{\frac{3}{2}}\log\epsilon\left[\frac{P_1}{8}YA^2(\cos 2n\pi z+1)\right] + \epsilon^{\frac{3}{2}}\left[\frac{U_c^{\text{iv}}}{5!}Y^5+\frac{\phi_c'''}{6}Y^3A\cos n\pi z+P_1 Y\log|Y|\frac{A^2}{4}(\cos 2n\pi z+1)+\ldots\right]+\ldots\right\}. \quad (3.38)$$

The equation governing the motion in the critical layer remains (3.10).

The solution for the first two terms in the expansion are given identically by the expressions in the matching condition. At the next order we have the equation

$$(S_Y\,\partial_\xi - S_\xi\,\partial_Y)(\Psi^{(2)}_{YY}+K^2\Psi^{(c)}_{ZZ}) = \lambda\Psi^{(2)}_{YYYY}. \quad (3.39)$$

Writing the function $\Psi^{(2)}$ as

$$\Psi^{(2)} = \Psi^{(2,0)}+\lambda\Psi^{(2,1)}+\ldots, \quad (3.40)$$

we obtain

$$\Psi^{(2)}_{YY} = F^{(2,0)}(S,T,Z)+k^2\phi_c A\cos n\pi z. \quad (3.41)$$

It is important to note that, although the solution $F^{(2,0)} = U_c''' U_c'^{-1}S$ satisfies the matching condition identically, it violates the secularity condition defined in (3.25), leading to a non-uniformity in the solution for the viscous correction $\Psi^{(2,1)}$. The secularity condition, as can be observed from (3.26), always leads to a homogeneous solution behaving asymptotically (for large Y) as $S^{\frac{1}{2}}$. The matching condition, on the other hand, requires a homogeneous solution behaving as S for large S. It appears that the difficulty can be circumvented only by introducing a viscous time scale in the critical layer. We point out that this difficulty is not unique to the long-wave analysis presented here, but, in fact, must appear in the problem discussed by Benney & Maslowe as well since the difficulty arises with the matching to the mean flow at successively higher orders.

Introducing a viscous time scale defined by

$$\vartheta = \delta t, \quad \delta = (R\epsilon^3)^{-1} = \lambda/\epsilon^{\frac{1}{2}}, \quad (3.42)$$

the equation for $\Psi^{(2,1)}$ becomes

$$\partial\Psi^{(2,1)}_{YY}/\partial X = -S_Y^{-1}F^{(2,0)}_{\vartheta}+(S_Y F^{(2,0)}_S)_S. \quad (3.43)$$

Applying the secularity condition now yields the partial differential equation for $F^{(2,0)}$:

$$\left(\frac{\Delta}{2\pi}\int_{-\pi/\Delta}^{\pi/\Delta}\frac{1}{S_Y}dX\right)F^{(2,0)}_{\vartheta} = \left\{F^{(2,0)}_S\left(\frac{\Delta}{2\pi}\int_{-\pi/\Delta}^{\pi/\Delta}S_Y\,dX\right)\right\}_S, \quad S > S_c. \quad (3.44)$$

This is a diffusion equation with variable coefficients. For $S < S_c$ we must take $F^{(2,0)}$ to be a constant as explained earlier. In order to show that the nonlinear critical layer can be used to construct a solution which is uniform for all y, we must demonstrate that

(3.44) admits solutions varying as S for large S. The same type of homogeneous equation, with corresponding higher-order viscous time scales, must be solved at all successive orders where mean-flow terms appear in the matching condition. For example, one can show quite readily, using the relations among various derivatives of the eigenfunction implied by the eigenvalue equation and after computing the particular solutions at each order, that the solution for $\Psi^{(4)}$ must behave as

$$\frac{U_c'''}{6}\left(\frac{U_c^{\text{iv}}}{U_c'''}-\frac{U_c''}{U_c'}\right)S^{\frac{3}{2}}$$

to satisfy the matching condition, $\Psi^{(6)}$ must vary as S^2, etc.

The general solution of (3.44) has not been obtained as yet. However, for present purposes it seems to be sufficient to show that the model equation

$$F_t = x^{\frac{1}{2}}(x^{\frac{1}{2}}F_x)_x \tag{3.45}$$

admits solutions varying as $x^{\frac{1}{2}n}$, $n = 2, 3, \ldots$, for x large so that the asymptotic matching conditions can be satisfied. To this end we examine the similarity solution of (3.45):

$$\left.\begin{aligned} F &= x^{\alpha} f(\eta), \quad \eta = x/t, \\ \eta^2 f'' + [\eta^2 &+ (2\alpha+\tfrac{1}{2})\,\eta]f' + \alpha(\alpha-\tfrac{1}{2})f = 0. \end{aligned}\right\} \tag{3.46}$$

One independent solution valid for large η behaves as

$$f = 1+\frac{\alpha(\alpha-\frac{1}{2})}{\eta}+\frac{\alpha(\alpha-\frac{1}{2})\,(2\alpha^2-5\alpha+3)}{\eta^2}+\ldots. \tag{3.47}$$

It is interesting to note that this series terminates after the η^{-1} term when $\alpha = 1$, after the η^{-2} term when $\alpha = 2$, etc. For half-integer values of α, all terms are present. The other independent solution decays exponentially for large η. Hence we conclude that application of the secularity condition, appropriately modified through the use of viscous time scales, and asymptotic matching yield a consistent description of the flow in the critical-layer region. Viscous boundary layers along the dividing streamlines may be required at various stages of the analysis if the vorticity is to be smoothly matched across the critical layer, but they cause no difficulty in principle.

4. Examples and discussion

We now present some solutions to the eigenvalue problems (2.14) and (2.31) and illustrate the kinds of streamline pattern associated with an isolated solitary wave that can appear for a couple of specific flows. In order to limit the discussion, we consider only RN modes and an asymmetric shear flow. There is no doubt that solutions to the eigenvalue problem for SN modes exist since Benney & Bergeron have computed them for several shear flows for the case $\beta = 0$, including the shear-layer case $U(y) = \tanh y$. If they exist for $\beta = 0$, there is ample reason to expect that they exist for $\beta \neq 0$.

For the KdV (homogeneous atmosphere) case, the RNM solution can be constructed for the shear flow

$$U(y) = \operatorname{sn}(y|m), \quad 0 < m < 1, \quad -\tfrac{1}{2}\pi < y < \tfrac{1}{2}\pi, \tag{4.1}$$

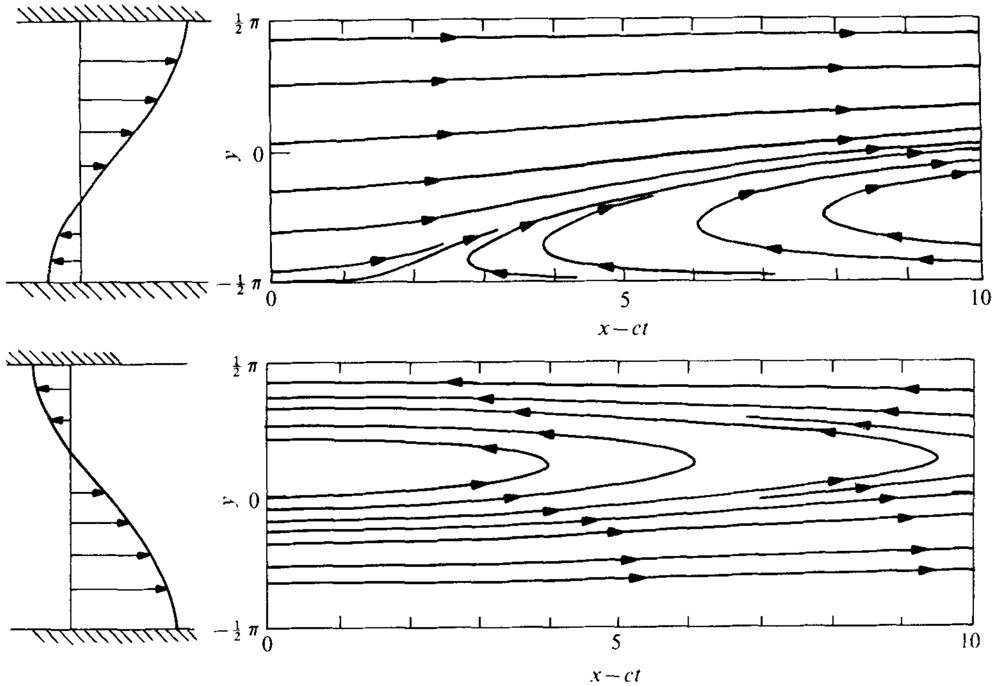

FIGURE 2. Streamline patterns for KdV solitons in an asymmetric shear flow; $m = 0{\cdot}1$, $\epsilon = 0{\cdot}4$.

where $\mathrm{sn}\,(y|m)$ is the Jacobian elliptic function with modulus m. The restriction $m > 0$ is necessary because the numerator of (2.34*a*) vanishes identically when $m = 0$. When m is small compared with unity, the solution for the first mode is given by

$$\phi(y) = \cos y + m\{\tfrac{1}{4}y \sin y + \tfrac{1}{16}\cos 3y - \tfrac{1}{3}c_0 \sin 2y\} + O(m^2), \tag{4.2}$$

where

$$c_0 = -\beta/(1+k_0^2) + O(m^2), \quad -1 < c_0 < 0. \tag{4.3}$$

The phase speed is westward (negative), consistent with the known property of Rossby waves. The coefficients (2.34) in the KdV equation for this case are evaluated to be

$$R_0 = -\frac{3m}{1+k_0^2}\frac{1}{c_0}(1 - 6c_0^2 + 4c_0^4), \quad S_0 = \frac{1}{2c_0(1+k_0^2)}. \tag{4.4}$$

It is interesting to note that R_0 changes sign along the eigenlocus, so that solitary waves of type E exist for $-0{\cdot}437 < c_0 < 0$ and solitary waves of type D exist for $-1 < c_0 < -0{\cdot}437$. These results are independent of the direction of the shear. Streamline patterns for each wave type are shown in figure 2. Solutions for higher modes and for other shear flows, including theoretical results and patterns for interactions of solitary waves with differing c_0, are being constructed and will be reported separately (Redekopp & Weidmann 1977).

When the atmosphere is stratified, a solution of the eigenvalue problem (2.14) exists for $m = 1$ ($U(y) = \tanh y$) and $-\infty < y < \infty$ in (4.1) and has been studied independently by Howard & Drazin (1964) and Lipps (1965). The eigenfunction was found to be

$$\phi_n(y) = (1+\tanh y)^{\frac{1}{2}(1-c_{0n})}(1-\tanh y)^{\frac{1}{2}(1+c_{0n})}, \tag{4.5}$$

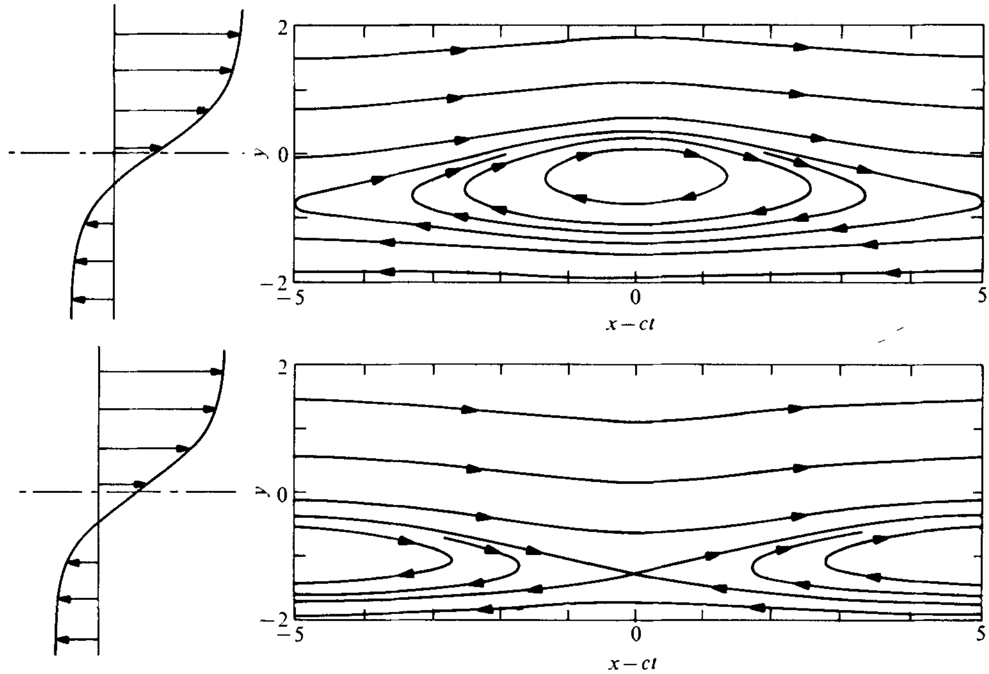

FIGURE 3. Streamline patterns for MKdV solitons in an unbounded asymmetric shear flow; $c_0 = -0{\cdot}7$, $\epsilon = 0{\cdot}4$.

with the eigenvalue relation

$$\beta = 2k_n^2(1-k_n^2)^{\frac{1}{2}} = -2c_{0n}(1-c_{0n}), \quad c_{0n} = -(1-k_n^2)^{\frac{1}{2}}. \tag{4.6}$$

A solution of the type given in (4.2) for $0 < m \ll 1$ can be constructed also, but it is, perhaps, more instructive to exhibit streamline patterns for the unbounded case of (4.5). Although the numerical computation of the coefficient R_n in the MKdV equation has not been accomplished for this case, it seems plausible from the solution for small m that points on the eigenlocus (4.6) exist for which R_n is positive. Assuming this to be true for $c_{01} = -0{\cdot}7$, we take $R_n = O(1)$ and compute the streamline patterns shown in figure 3. Note that we have chosen the amplitude to be quite large ($\epsilon = 0{\cdot}4$) in order to dramatize the kind of pattern that emerges in an extreme limit of the theory.

The latter eigenvalue solution was included because it provides a convenient basis for presenting a conjecture about the 'permanence' of such entities, even in a slightly dissipative fluid. The conjecture hinges on the relationship of the solution (4.5) to the linear stability of the shear flow. In the eigenvalue relation (4.6), which is depicted in figure 4, the atmospheric parameter k_n takes the usual role of the wavenumber in stability theory. The true wavenumber is $O(\epsilon)$ ($O(\epsilon^{\frac{1}{2}})$ for the KdV case) and does not enter at this stage of approximation, but the implications of including it are interesting. Linear stability theory (cf. Lipps 1965) shows that values of (β, k_n^2) under the neutral eigenlocus lead to growing modes (instability) and values above the curve lead to damped modes (stability). Now, if the wavenumber α, say, were included in (4.6), the neutral curve in figure 4 would be the same, except that the abscissa would be $k_n^2 + \alpha^2$. This implies that those waves corresponding to points on the eigenlocus with $k_n^2 > k_n^2(\beta_{\max}) = \frac{2}{3}$ would experience a small growth. That is, they could continually,

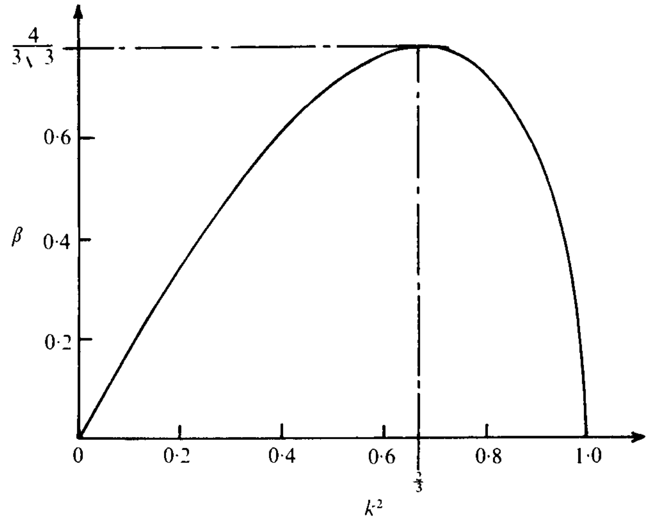

FIGURE 4. The neutral curve for regular neutral modes in $U(y) = \tanh y$ shear flow.

albeit slowly, extract energy from the mean shear flow to sustain their shape and motion. A preliminary attempt to quantify this conjecture is pursued in the following section.

5. Higher-order evolution properties

Previously we found that the amplitude evolution of a long Rossby wave in a stratified atmosphere was governed, to leading order, by the MKdV equation. That equation admitted stable, permanent, solitary-wave solutions. In this section we obtain the next-order corrections to the MKdV equation with a view to ascertaining whether the higher-order terms will reflect the growth mechanism conjectured in the previous section. To continue the previous analysis to higher order, we follow a perturbation scheme in which we continue to assume that A depends only on the slow space and time variables (ξ, τ), but that the evolution equation (2.20) will have to be modified appropriately at successively higher orders.

In § 2.1 we found that the first three terms in the expansion for the perturbation stream function (2.10) were given by

$$\psi^{(1)} = A(\xi, \tau)\,\phi(y)\cos n\pi z, \tag{5.1}$$

$$\psi^{(2)} = \tfrac{1}{4}A^2(\phi^{(2,2)}\cos 2n\pi z + \phi^{(2,0)}) \tag{5.2}$$

and

$$\psi^{(3)} = A_{\xi\xi}\,\phi_1^{(3,1)}\cos n\pi z + A^3\phi_2^{(3,1)}\cos n\pi z + A^3\phi^{(3,3)}\cos 3n\pi z, \tag{5.3}$$

provided that $A(\xi, \tau)$ obeyed the MKdV equation. Continuing the analysis to higher orders one finds, after some effort, that

$$\begin{aligned}\psi^{(4)} = {} & A^4\{\phi_1^{(4,4)}(y)\cos 4n\pi z + \phi_1^{(4,2)}(y)\cos 2n\pi z + \phi_1^{(4,0)}(y)\} \\ & + (A_\xi)^2\{\phi_2^{(4,4)}\cos 4n\pi z + \phi_2^{(4,2)}\cos 2n\pi z + \phi_2^{(4,0)}\} \\ & + (A^2)_{\xi\xi}\{\phi_3^{(4,4)}\cos 4n\pi z + \phi_3^{(4,2)}\cos 2n\pi z + \phi_3^{(4,0)}\},\end{aligned} \tag{5.4}$$

and that the contribution to $\psi^{(5)}$ from the nth internal-wave mode, $\psi^{(5,1)}$ say, is given by

$$\psi^{(5,1)} = \{A^5\phi_1^{(5,1)}(y) + (A^3)_{\xi\xi}\phi_2^{(5,1)}(y) + A(A_\xi)^2\phi_3^{(5,1)}(y) + A^2A_{\xi\xi}\phi_4^{(5,1)}(y) + A_{\xi\xi\xi\xi}\phi_5^{(5,1)}(y)\}\cos n\pi z. \quad (5.5)$$

The equations defining the various functions of y are lengthy and will be omitted. The last result also requires the evolution equation to assume the form

$$A_\tau + \tfrac{1}{3}R_n(A^3)_\xi + S_nA_{\xi\xi\xi} = \epsilon^2\{\nu_{1n}A_{\xi\xi\xi\xi\xi} + \nu_{2n}(A^2A_{\xi\xi})_\xi + \nu_{3n}[A(A_\xi)^2]_\xi + \nu_{4n}(A^3)_{\xi\xi\xi} + \nu_{5n}(A^5)_\xi\} + O(\epsilon^4). \quad (5.6)$$

The coefficients ν_{in} are determined in the same manner as that outlined in § 2.1. It is evident that the higher-order terms are dispersive and do not reflect any growth mechanism. In fact, the equation admits a solitary-wave solution of the type

$$A = (A_0 + \epsilon^2A_2\operatorname{sech}^2\theta)\operatorname{sech}\theta, \quad \theta = \theta_0 + \epsilon^2\theta_2, \quad (5.7)$$

showing that the amplitude, shape and speed are changed to $O(\epsilon^2)$. It appears, therefore, that a weakly growing solitary-wave theory, at least in the long-wave limit in mind here, must incorporate the effect of viscosity in a more direct way. The viscous time scale in the critical layer may be influential to this end, but we have not succeeded in delineating its role in the wave evolution as yet. However, if we simply allow the coefficients in the MKdV equation to have small imaginary parts consistent with a viscous critical-layer approach and then perform a perturbation analysis using the soliton as the lowest-order solution, one is able to show that the soliton can persist even in the presence of a small damping effect. This work is still in progress.

6. Concluding remarks

We have examined some dynamical and mathematical aspects of solitary Rossby waves and have shown that solitary waves of various morphologies exist in a shear flow. The flow pattern associated with the E-soliton type delineated herein has a form strikingly similar to observations of the Great Red Spot on Jupiter. A preliminary discussion of how the present theory may apply (e.g. length scales, atmospheric parameters, etc.) in that case has been presented by Maxworthy & Redekopp (1976*a*, *b*). Some appealing aspects of a soliton model for the Red Spot, which are lacking in all other hypotheses concerning its fluid-dynamic character, include the retention of identity following interactions with other (soliton) features, the permanence of the wave form, and the fact that the shape and the propagation speed will vary in a unique way depending on the sheared zonal current. Further model studies and an intensive review of Jovian observations are underway to help scrutinize this model.

It is anticipated that certain features of solitary Rossby waves may emerge as important descriptive entities in the terrestrial atmosphere and ocean as well. A necessary requirement for the existence of these waves is that shear flows persist for times long enough for the formation of solitons. This time is quite long ($O(\epsilon^{-3})$ for the MKdV case and $O(\epsilon^{-\frac{3}{2}})$ for the KdV case). However, the important point we wish to emphasize is that the present analysis demonstrates the existence of coherent eddy motions with very peculiar interaction properties and a 'permanent' identity in marginally stable shear flows. The same kind of theory is being developed for internal waves in shear flows and will be reported separately. In fact, the types of solitary-wave solutions presented here are not complete. The MKdV equation also admits unsteady, localized, solitons called 'breathers' and some interesting examples of these motions are presented in Redekopp & Weidman. The distinction between solitons with critical layers and the large-scale features of turbulent motion in observations or measurements in these kinds of flows, then, is not so clear. Indeed, the dynamics and existence of solitary waves in shear flows in general deserve a closer look.

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